Table 1. *Minimal balance surfaces built up from branched catenoids*

Minimal balance surface	Group–subgroup pair	Genus	Surface patches: Point group	Surface patches: Generating circuits	Number of equivalent surfaces	Transformations
$BC1$	$P6_322$–$P6_3$	9	3..	3+9: $000, 100, 110/\frac{2}{3}\frac{1}{3}\frac{1}{4}, 00\frac{1}{4}, \frac{1}{3}\frac{\bar{1}}{3}\frac{1}{4}, \frac{2}{3}\frac{1}{3}\frac{1}{4}, 10\frac{1}{4}, \frac{4}{3}\frac{2}{3}\frac{1}{4}, \frac{2}{3}\frac{1}{3}\frac{1}{4}, 11\frac{1}{4}, \frac{1}{3}\frac{2}{3}\frac{1}{4}$	4	$m(xy0)$; $m(2x, x, z)$; $2(2x, x, z)$
$BC2$	$P4_2/nnm$–$P4_2nm$	7	$2.mm$	4+8: $\frac{1}{2}0\frac{1}{4}, 0\frac{1}{2}\frac{1}{4}, \frac{\bar{1}}{2}0\frac{1}{4}, 0\frac{1}{2}\frac{1}{4}/000, \frac{1}{2}00, \frac{1}{2}\frac{1}{2}0, 0\frac{1}{2}0, 000, \frac{\bar{1}}{2}00, \frac{\bar{1}}{2}\frac{\bar{1}}{2}0, 0\frac{\bar{1}}{2}0$	4	$m(xy\frac{1}{4})$; $m(0yz)$; $2(0y\frac{1}{4})$
$BC3$	$I422$–$I4$	6	4..	4+12: $\frac{1}{2}0\frac{1}{4}, 0\frac{1}{2}\frac{1}{4}, \frac{\bar{1}}{2}0\frac{1}{4}, 0\frac{\bar{1}}{2}\frac{1}{4}/000, \frac{1}{2}\frac{1}{2}0, 0\frac{1}{2}0, 000, \frac{\bar{1}}{2}\frac{1}{2}0, \frac{\bar{1}}{2}00, 000, \frac{\bar{1}}{2}\frac{\bar{1}}{2}0, 0\frac{\bar{1}}{2}0, 000, \frac{1}{2}\frac{\bar{1}}{2}0, \frac{1}{2}00$	4	$m(xy\frac{1}{4})$; $m(0yz)$; $2(0y\frac{1}{4})$

Information on the properties of BC surfaces is summarized in Table 1. In each case, for one of the four equivalent surfaces a pair of generating circuits is described by its vertices. Generating circuits for the other three surfaces may be calculated with the aid of the symmetry operations listed in the last column.

Minimal surfaces of the families $BC1$, $BC2$ and $BC3$ are not complementary to surfaces of other families described so far. In a subsequent paper, however, a family of minimal surfaces complementary to the $BC2$ surfaces will be presented.

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New Surface Patches for Minimal Balance Surfaces. II. Multiple Catenoids

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Abstract

Eight new families of minimal balance surfaces are described. Their surface patches belong to a new kind, called multiple catenoids. The generating circuits of such a minimal surface are two congruent concave polygons with one point of self-contact each. The new minimal balance surfaces are complementary to other minimal balance surfaces which are built up from catenoid-like surface patches and have been known before.

1. Introduction

The symmetry of each minimal balance surface can be described by a group–subgroup pair $G \supset H$ of space groups with index 2, its inherent symmetry. The fixed points of all symmetry operations s with $s \in G$ but $s \notin H$ are necessarily contained within the surface (Fischer & Koch, 1987). Most of the minimal balance surfaces described so far have a linear skeletal net, *i.e.* a set of twofold axes defined by the corresponding space-group pair, that is embedded within the surface (*cf.* Schoen, 1970; Hyde & Andersson, 1984). Such a set of twofold axes may be used to generate a minimal balance surface (*cf.* Fischer & Koch, 1987, 1989; Koch & Fischer, 1988). Then it is called a generating linear net.

As all sets of twofold axes defined by space-group pairs with index 2 may be assigned to 52 cases (*cf.* Koch & Fischer, 1988, Table 1) at most 52 types of generating linear nets for minimal balance surfaces exist. Such a set of twofold axes may be three-dimensionally connected or not. Among the disconnected sets those ones stand out that disintegrate into parallel nets.

If all nets of such a set are congruent [cases 22 to 30 in Table 1 of Koch & Fischer (1988)] catenoid-like

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surface patches may be constructed in most cases (22 to 28). The corresponding minimal surfaces have been derived completely (Schwarz, 1890; Schoen, 1970; Koch & Fischer, 1988). If nets of two different kinds are stacked alternately within the same set of twofold axes branched catenoids may be constructed. The three corresponding families of minimal surfaces are described by Fischer & Koch (1989).

Within the present paper eight new families of minimal balance surfaces will be presented, the surface patches of which are multiple catenoids.

2. Multiple catenoids

Catenoid-like surface patches of minimal balance surfaces are bounded by two parallel congruent flat and convex polygons, branched catenoids by two different parallel flat polygons, one of which is convex whereas the other one is concave with one point of self-contact.

A *multiple catenoid* may be imagined as resulting from fusion of n neighbouring catenoids. These n catenoids are bounded by two times n convex polygons that belong to a pair of adjacent nets of twofold axes. The n polygons within one of these nets share a common vertex. Accordingly, two parallel congruent flat and concave polygons may be formed with one point of self-contact each (*cf.* Figs. 1 to 9). These two concave polygons are the boundaries of a multiple catenoid.

Multiple catenoids are compatible with all sets of congruent parallel nets of twofold axes stacked directly upon each other (cases 22 to 26). They are incompatible with sets of congruent nets where vertices of one net lie above the polygon centres of another net (cases 27 to 30) and with sets of parallel nets of two different kinds (cases 31 to 33).

The triangular nets of twofold axes referring to cases 23 (46 . 12) and 25 (48^2) contain inequivalent kinds of vertices. Accordingly, different types of multiple catenoids may be formed.

3. Minimal balance surfaces *MC*1 (triple catenoids)

Sets of twofold axes that disintegrate into parallel nets 3^6 of equilateral triangles belong to case 22. They allow the construction of triple catenoids with point symmetry $\bar{6}2m$ bounded by two concave 9-gons (Fig. 1). These triple catenoids are the surface patches for a family of minimal balance surfaces with inherent symmetry $P6_3/mcm$–$P\bar{6}2m$ (*cf.* Table 1) designated *MC*1. As the generating linear nets described above form also the linear skeletal nets of these surfaces, *H* surfaces (*cf.* Schoen, 1970; Koch & Fischer, 1988) and *MC*1 surfaces are complementary.

Each *MC*1 surface consists of triple catenoids in two different orientations. All triple catenoids from the same layer, *i.e.* triple catenoids bounded by the same pair of adjacent nets 3^6, are in parallel orientation. Triple catenoids from different layers are oriented differently. The genus of an *MC*1 surface equals 7.

Case 22 refers to group-subgroup pairs *G*–*H* of 18 different types. Eight of them are incompatible with *MC*1 surfaces for the same reasons as they are incompatible with *H* surfaces (*cf.* Koch & Fischer, 1988). For five of the other ten types the triangular nets are formed by axes .2. such that all vertices within a net are symmetrically equivalent. As a consequence, the combination of three triangles to a concave 9-gon is impossible without symmetry reduction. The remaining five types are $P6_3/mcm$–$P\bar{6}2m$, $P\bar{6}c2$–$P\bar{6}$, $P6_322$–$P321$, $P\bar{3}1m$–$P31m$ and $P312$–$P3$.

Each set of twofold axes built up from triangular nets 3^6 may form the generating linear net of six congruent and complementary *MC*1 surfaces. If any concave 9-gon has been chosen as the first boundary of a triple catenoid the second one may belong to the triangular net above or below. The corresponding two *MC*1 surfaces show identical inherent symmetry. The central axes of their triple catenoids coincide. Both surfaces are mapped onto each other by a symmetry operation of the intersection group $N_E(P6_3/mcm) \cap N_E(P\bar{6}2m) = P6/mmm(c/2)$ of the Euclidean normalizers of *G* and *H*, *e.g.* by a mirror reflection *m*.. (*cf.* Table 1, column 'Transformations'). In addition, there exist two further possibilities to choose the central axes of the triple catenoids. Each of these choices refers to two other congruent *MC*1 surfaces and to another space-group pair $P6_3/mcm$–$P\bar{6}2m$ which is shifted against the original one by a vector $(\frac{2}{3}\frac{1}{3}0)$ or $(\frac{1}{3}\frac{2}{3}0)$ (*cf.* column 'Origin shifts' of Table 1).

4. Minimal balance surfaces *MC*2, *MC*3 and *MC*4 (double, triple and sextuple catenoids)

Case 23 describes sets of twofold axes that disintegrate into triangular nets 46 . 12 (angles 30, 60, 90°).

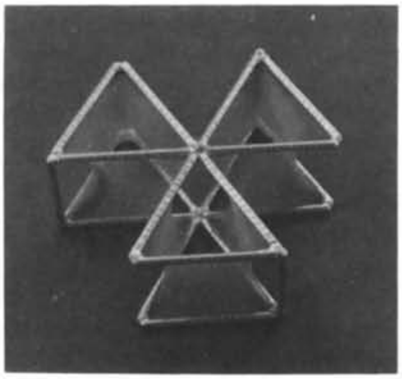

Fig. 1. Triple catenoid, a surface patch of a minimal surface *MC*1 with symmetry $P6_3/mcm$–$P\bar{6}2m$.

Table 1. *Minimal balance surfaces built up from multiple catenoids*

The coordinates of the vertices are given for one of the generating circuits only. Those of the second one result from adding $00\frac{1}{2}$. Column 'Transformations' refers to congruent complementary surfaces with identical inherent symmetry, column 'Origin shifts' to congruent complementary surfaces with different inherent symmetry.

Minimal balance surface	Group-subgroup pair	Nets of twofold axes	Genus	Surface patches: Point group	Surface patches: Generating circuits	Number of congruent surfaces	Transformations	Origin shifts
$MC1$	$P6_3/mcm$-$P\bar{6}2m$	6^3	7	$\bar{6}2m$	9+9: $000, \frac{2}{3}\frac{1}{3}0, \frac{1}{3}\frac{2}{3}0, 000, \frac{\bar{1}}{3}\frac{1}{3}0, \frac{\bar{2}}{3}\frac{\bar{1}}{3}0, 000, \frac{\bar{1}}{3}\frac{\bar{2}}{3}0, \frac{1}{3}\frac{\bar{1}}{3}0$	3×2	$m(xy0)$	$\frac{2}{3}\frac{1}{3}0$; $\frac{1}{3}\frac{2}{3}0$
$MC2$	$P6/mcc$-$P6/m$	46 . 12	13	$2/m..$	6+6: $\frac{1}{2}\frac{1}{2}\frac{1}{4}, \frac{1}{3}\frac{2}{3}\frac{1}{4}, 00\frac{1}{4}, \frac{1}{2}\frac{1}{2}\frac{1}{4}, \frac{2}{3}\frac{1}{3}\frac{1}{4}, 11\frac{1}{4}$	2	$m(xy\frac{1}{4})$	
$MC3$	$P6/mcc$-$P6/m$	46 . 12	13	$\bar{6}..$	9+9: $\frac{2}{3}\frac{1}{3}\frac{1}{4}, 00\frac{1}{4}, \frac{1}{2}0\frac{1}{4}, \frac{2}{3}\frac{1}{3}\frac{1}{4}, 10\frac{1}{4}, 1\frac{1}{2}\frac{1}{4}, \frac{2}{3}\frac{1}{3}\frac{1}{4}, 11\frac{1}{4}, \frac{1}{2}\frac{1}{2}\frac{1}{4}$	2	$m(xy\frac{1}{4})$	
$MC4$	$P6/mcc$-$P6/m$	46 . 12	13	$6/m..$	18+18: $00\frac{1}{4}, \frac{1}{2}0\frac{1}{4}, \frac{2}{3}\frac{1}{3}\frac{1}{4}, 00\frac{1}{4}, \frac{1}{2}\frac{1}{2}\frac{1}{4}, \frac{1}{3}\frac{2}{3}\frac{1}{4}, 00\frac{1}{4}, 0\frac{1}{2}\frac{1}{4}, \frac{\bar{1}}{3}\frac{1}{3}\frac{1}{4}, 00\frac{1}{4}, \frac{\bar{1}}{2}0\frac{1}{4}, \frac{\bar{2}}{3}\frac{\bar{1}}{3}\frac{1}{4}, 00\frac{1}{4}, \frac{\bar{1}}{2}\frac{\bar{1}}{2}\frac{1}{4}, \frac{\bar{1}}{3}\frac{\bar{2}}{3}\frac{1}{4}, 00\frac{1}{4}, 0\frac{\bar{1}}{2}\frac{1}{4}, \frac{1}{3}\frac{\bar{1}}{3}\frac{1}{4}$	2	$m(xy\frac{1}{4})$	
$MC5$	$P4_2/mcm$-$Cmmm$	4^4	5	$m.mm$	8+8: $00\frac{1}{4}, \frac{1}{2}0\frac{1}{4}, \frac{1}{2}\frac{1}{2}\frac{1}{4}, 0\frac{1}{2}\frac{1}{4}, 00\frac{1}{4}, \frac{\bar{1}}{2}0\frac{1}{4}, \frac{\bar{1}}{2}\frac{\bar{1}}{2}\frac{1}{4}, 0\frac{\bar{1}}{2}\frac{1}{4}$	2×4	$m(xy\frac{1}{4})$; $t(\frac{1}{2}\frac{1}{2}0)$; $n(xy\frac{1}{4})$	$\frac{1}{2}00$
$MC6$	$I4/mcm$-$P4/mbm$	48^2	9	$m.mm$	6+6: $\frac{1}{2}0\frac{1}{4}, \frac{1}{2}\frac{1}{2}\frac{1}{4}, 00\frac{1}{4}, \frac{1}{2}0\frac{1}{4}, \frac{1}{2}\frac{\bar{1}}{2}\frac{1}{4}, 10\frac{1}{4}$	2	$m(xy\frac{1}{4})$	
$MC7$	$P4/mcc$-$P4/m$	48^2	9	$4/m..$	12+12: $00\frac{1}{4}, \frac{1}{2}0\frac{1}{4}, \frac{1}{2}\frac{1}{2}\frac{1}{4}, 00\frac{1}{4}, 0\frac{1}{2}\frac{1}{4}, \frac{\bar{1}}{2}\frac{1}{2}\frac{1}{4}, 00\frac{1}{4}, \frac{\bar{1}}{2}0\frac{1}{4}, \frac{\bar{1}}{2}\frac{\bar{1}}{2}\frac{1}{4}, 00\frac{1}{4}, 0\frac{\bar{1}}{2}\frac{1}{4}, \frac{1}{2}\frac{\bar{1}}{2}\frac{1}{4}$	4	$m(xy\frac{1}{4})$; $t(\frac{1}{2}\frac{1}{2}0)$; $n(xy\frac{1}{4})$	
$oMC5$	$Pccm$-$P2/m$	4^4	5	$..2/m$	8+8: $00\frac{1}{4}, \frac{1}{2}0\frac{1}{4}, \frac{1}{2}\frac{1}{2}\frac{1}{4}, 0\frac{1}{2}\frac{1}{4}, 00\frac{1}{4}, \frac{\bar{1}}{2}0\frac{1}{4}, \frac{\bar{1}}{2}\frac{\bar{1}}{2}\frac{1}{4}, 0\frac{\bar{1}}{2}\frac{1}{4}$	8	$m(xy\frac{1}{4})$; $t(\frac{1}{2}\frac{1}{2}0)$; $n(xy\frac{1}{4})$; $t(\frac{1}{2}00)$; $a(xy\frac{1}{4})$; $t(0\frac{1}{2}0)$; $b(xy\frac{1}{4})$	

The corresponding group-subgroup pairs belong to the types $P6/mcc$-$P6/m$, $P622$-$P6$, and $P622$-$P622(2c)$. Only the first two of these types are compatible with catenoid-like surface patches and with the corresponding minimal balance surfaces of the family $R3$ (*cf.* Schoen, 1970; Koch & Fischer, 1988). They are compatible in addition with minimal surfaces built up from multiple catenoids.

As in a triangular net 46 . 12 three kinds of vertices exist shared by four, six and twelve triangles, respectively, three kinds of multiple catenoids may be formed. Two triangles may be combined to a concave 6-gon (with self-contact at the 90° vertex), three triangles to a concave 9-gon (with self-contact at the 60° vertex), or six triangles to a concave 18-gon (with self-contact at the 30° vertex). Accordingly, the fusion of two, three or six catenoids results in a double, triple or sextuple catenoid (*cf.* Figs. 2 to 5). The corresponding minimal balance surfaces are designated $MC2$, $MC3$ and $MC4$, respectively.

The double catenoids of an $MC2$ surface show six different orientations, the triple catenoids of an $MC3$ surface four, and the sextuple catenoids of an $MC4$ surface two different orientations. Multiple catenoids from adjacent layers are always oriented differently; within a single layer three orientations occur for the double catenoids, two orientations for the triple catenoids and one orientation only for the sextuple catenoids.

The genus of all these minimal surfaces equals 13. Their inherent symmetry is the same as for $R3$ surfaces, namely $P6/mcc$-$P6/m$. Sets of twofold axes belonging to case 23 may be used as generating linear nets for $R3$ surfaces as well as for $MC2$, $MC3$ and $MC4$ surfaces. These generating linear nets are also the linear skeletal nets of all these surfaces and, as a consequence, all these minimal surfaces are complementary.

According to the intersection group $N_E(P6/mcc) \cap N_E(P6/m) = P6/mmm(c/2)$ each

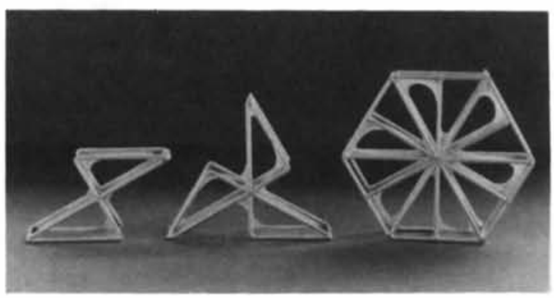

Fig. 2. The different multiple catenoids spanned between two nets 46 . 12.

set of twofold axes belonging to case 23 is compatible with two congruent surfaces of each of the families $MC2$, $MC3$ and $MC4$, *i.e.* with six surfaces built up from multiple catenoids. Two congruent surfaces can be mapped onto another by a mirror reflection $m..$ belonging to $P6/mmm(c/2)$.

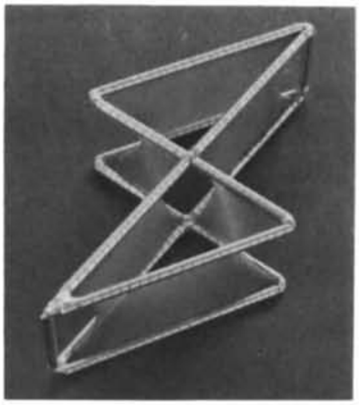

Fig. 3. Double catenoid, a surface patch of a minimal surface $MC2$ with symmetry $P6/mcc$-$P6/m$.

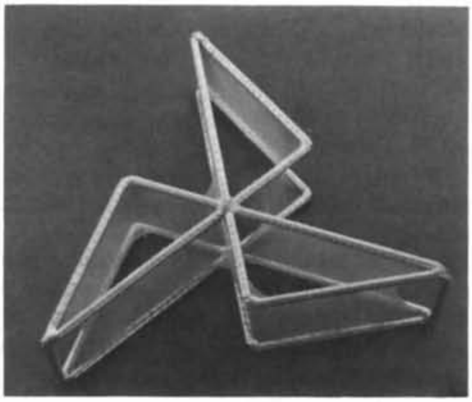

Fig. 4. Triple catenoid, a surface patch of a minimal surface $MC3$ with symmetry $P6/mcc$-$P6/m$.

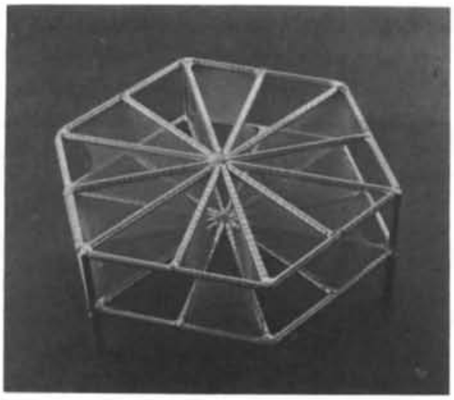

Fig. 5. Sextuple catenoid, a surface patch of a minimal surface $MC4$ with symmetry $P6/mcc$-$P6/m$.

5. Minimal balance surfaces $MC5$ (double catenoids)

Case 24 refers to square nets 4^4 of twofold axes. Such sets of twofold axes are defined by group-subgroup pairs of 46 types [*cf.* Table 1 of Koch & Fischer (1988)]. 15 of these types are compatible with tetragonal distorted P surfaces tP made up from catenoids bounded by two squares. Fusion of two such catenoids results in double catenoids, the surface patches of $MC5$ surfaces. Such a double catenoid is bounded by two concave 8-gons with one point of self-contact each (*cf.* Fig. 6).

$MC5$ surfaces are compatible with group-subgroup pairs of only three out of the 15 types mentioned above: $P4_2/mcm$-$Cmmm$, $P\bar{4}2m$-$Cmm2$, and $P4_222$-$C222$. In all three cases the square nets are formed by rotation axes .2., *i.e.* the nets are oriented parallel to the coordinate axes.

The inherent symmetry of $MC5$ surfaces is $P4_2/mcm$-$Cmmm$ and the generating linear nets of these surfaces are also their linear skeletal nets. $MC5$ surfaces are complementary to tP surfaces though the inherent symmetries of such complementary surfaces differ. Exceptions are those $MC5$ surfaces which correspond to the limiting case of tP surfaces with cubic symmetry ($a = 2^{1/2}c$). As cubic P surfaces contain more twofold axes than tP surfaces, $MC5$ surfaces with $a = 2^{1/2}c$ are not complementary to P surfaces.

Each $MC5$ surface consists of double catenoids in two orientations. Double catenoids of the same layer are in parallel orientation, those of adjacent layers are in different orientation. The genus of $MC5$ surfaces equals 5.

Each set of square nets of twofold aces (case 24) is compatible with eight congruent $MC5$ surfaces. According to the intersection group of the two Euclidean normalizers $N_E(P4_2/mcm) \cap N_E(Cmmm) = P4/mmm([a-b]/2, [a+b]/2, c/2)$, four of these $MC5$ surfaces coincide in their inherent symmetry. The corresponding symmetry operations,

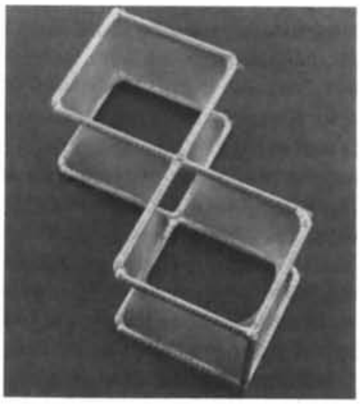

Fig. 6. Double catenoid, a surface patch of a minimal surface $MC5$ with symmetry $P4_2/mcm$-$Cmmm$.

that map the first $MC5$ surface onto the three equivalent ones, are listed in Table 1. The four other $MC5$ surfaces with the same linear skeletal net belong to a pair $P4_2/mcm$–$Cmmm$ which is shifted against the original pair by a vector $(\frac{1}{2}00)$ (*cf.* Table 1).

6. Minimal balance surfaces *MC*6 and *MC*7 (double and quadruple catenoids)

Group-subgroup pairs of nine different types define triangular nets 48^2 of twofold axes with tetragonal symmetry (angles 45, 45, 90°). Such sets of twofold axes [case 25 in Table 1 of Koch & Fischer (1988)] form the generating linear nets for $R2$ surfaces (Schoen, 1970) made up from catenoid-like surface patches. $R2$ surfaces are compatible with the following five types of group-subgroup pairs: $I4/mcm$–$P4/mbm$, $I422$–$P42_12$, $P4/mcc$–$P4/m$, $P4/nbm$–$P4bm$, and $P422$–$P4$.

The catenoids of an $R2$ surface may be combined in pairs to double catenoids (*cf.* Figs. 7, 8). These are bounded by two concave 6-gons which may be constructed from two triangles with a common 90° vertex. Such double catenoids occur as surface patches in two different orientations in minimal balance surfaces designated $MC6$. The double catenoids of each layer belong to both orientations and those from neighbouring layers with the same middle axes show different orientations.

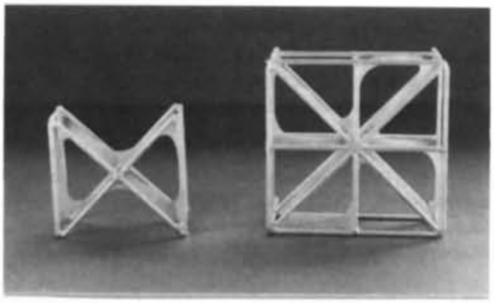

Fig. 7. The different multiple catenoids spanned between two nets 48^2.

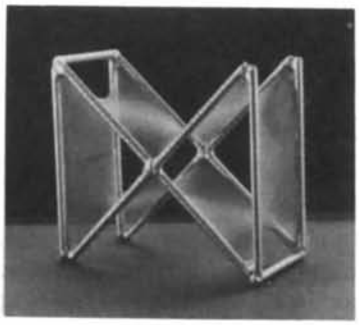

Fig. 8. Double catenoid, a surface patch of a minimal surface $MC6$ with symmetry $I4/mcm$–$P4/mbm$.

$MC6$ surfaces and $R2$ surfaces are compatible with group-subgroup pairs of the same types. The inherent symmetry is $I4/mcm$–$P4/mbm$ in both cases and the generating linear nets coincide with their linear skeletal nets.

Each set of 48^2 nets of twofold axes forms the generating linear net of two congruent $MC6$ surfaces (as well as of two $R2$ surfaces). According to the intersection group of the two Euclidean normalizers $N_E(I4/mcm) \cap N_E(P4/mbm) = P4/mmm([a-b]/2, [a+b]/2, c/2)$, the two congruent minimal surfaces show the same inherent symmetry. They are mapped onto another by a reflection $m..$ out of this intersection group.

The catenoids of an $R2$ surface may also be combined to quadruple catenoids (Figs. 7, 9). These are bounded by concave 12-gons with one point of self-contact. Each 12-gon consists of four triangles which meet in a common 45° vertex. Like the double catenoids the quadruple catenoids occur in two orientations in the corresponding minimal surfaces $MC7$. But in contrast to the double catenoids in $MC6$ surfaces all quadruple catenoids within the same layer of an $MC7$ surface are in parallel orientation. Quadruple catenoids from adjacent layers are oriented differently.

As $MC7$ surfaces contain fewer translations than $MC6$ or $R2$ surfaces they are compatible with only two of the five types of group-subgroup pairs listed above, namely $P4/mcc$–$P4/m$ and $P422$–$P4$. Though the inherent symmetry $P4/mcc$–$P4/m$ of $MC7$ surfaces differs from that of $MC6$ or $R2$ surfaces, surfaces of all three families are complementary. The generating linear nets of the $MC7$ surfaces are also their linear skeletal nets.

In contrast to the $MC6$ surfaces each set of 48^2 nets of twofold axes gives rise to four congruent $MC7$ surfaces. All four surfaces have identical inherent symmetry as may be learned from the intersection group $N_E(P4/mcc) \cap N_E(P4/m) = P4/mmm([a-$

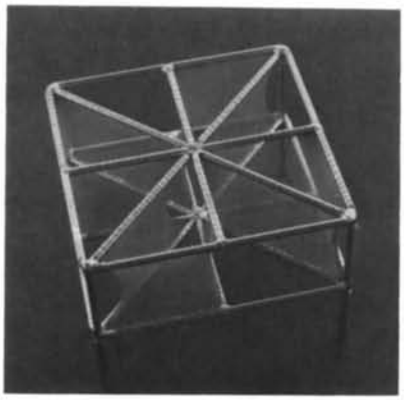

Fig. 9. Quadruple catenoid, a surface patch of a minimal surface $MC7$ with symmetry $P4/mcc$–$P4/m$.

$b]/2$, $[a+b]/2$, $c/2$). These four surfaces are mapped onto another, for example, by the symmetry operations listed in Table 1.

The genus is 9 for $MC6$ as well as for $MC7$ surfaces.

7. Minimal surfaces $oMC5$ (double catenoids)

Rectangular nets 4^4 of twofold axes (case 26) are defined by orthorhombic group-subgroup pairs of 33 types. Only 12 of them are compatible with catenoid-like surface patches and with the respective minimal balance surfaces *oP*, a family of orthorhombically distorted *P* surfaces.

Quite similarly as described above for $MC5$ surfaces, the fusion of two such catenoids results in a double catenoid, a surface patch of an $oMC5$ surface. The surfaces of the family $oMC5$ may be regarded as orthorhombically distorted $MC5$ surfaces. The family $oMC5$ comprises the surfaces of the family $MC5$ as a limiting case ($a=b$).

$oMC5$ surfaces are compatible with group-subgroup pairs of only two of the 12 types mentioned above: $Pccm$-$P2/m$ and $P222$-$P2$. The inherent symmetry of $oMC5$ surfaces is $Pccm$-$P2/m$.

Analogously to $MC5$ surfaces each set of rectangular nets of twofold axes may generate eight congruent $oMC5$ surfaces; but in contrast to $MC5$ surfaces all these $oMC5$ surfaces have identical inherent symmetry. They can be mapped onto another by symmetry operations of the intersection group $N_E(Pccm) \cap N_E(P2/m) = Pmmm(a/2,\ b/2,\ c/2)$ (*cf.* Table 1).

8. Common properties of *MC* surfaces

For all minimal balance surfaces built up from multiple catenoids two layers of such catenoids exist per c-translation period. The central axes of the multiple catenoids are the same for the catenoids of different layers. Multiple catenoids from different layers with the same central axis are oriented differently.

If the generating linear net of an *MC* surface consists of triangular nets of twofold axes six MC surfaces exist which are complementary to each other. In the case of quadrangular nets eight complementary *MC* surfaces occur. Each vertex of a triangle or a quadrangle corresponds to two congruent MC surfaces. Equivalent vertices give rise to congruent surfaces, non-equivalent ones to non-congruent surfaces. Each of these surfaces is complementary in addition to two congruent minimal surfaces built up from catenoids (except $MC5$ surfaces with $a=2^{1/2}c$ and $oMC5$ surfaces with $a=b=2^{1/2}c$). The use of the capital letter C for the designation of complicated new minimal surfaces which are complementary to known ones is therefore misleading [*cf.* Schoen (1970): $C(H)$, $C(P)$, $C(D)$; Fischer & Koch (1987): $C(S)$, $C(Y)$ *etc.*], and should be avoided in the future.

Minimal surfaces with multiple catenoids as surface patches exist only within a certain range of the axial ratio $0<c/a\le c/a(\text{max.})$. As for minimal surfaces consisting of catenoids or branched catenoids the upper limits $c/a(\text{max.})$ are unknown. It has been shown by soap-film experiments that multiple catenoids allow a larger distance between neighbouring nets of twofold axes than the corresponding simple catenoids.

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Bloch-Wave Solution in the Bragg Case

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Abstract

The Bloch-wave method for reflection diffraction problems, primarily electron diffraction as in reflection high-energy electron diffraction (RHEED) and reflection electron microscopy (REM), is developed. The basic Bloch-wave approach for surfaces is reviewed, introducing the current flow concept which plays a major role both in understanding reflection diffraction and determining the allowed Bloch waves. This is followed by a brief description of the numerical methods for obtaining the results including

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